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# Buffering of Slow Terminals

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*In a previous paper, a model for queuing processes with correlated inputs was analyzed. To illustrate the application of those results, we model a concentrator of slow terminals. The sources of data, the terminals, generate data at a slower rate than the output speed of the buffer. In special cases, we obtain closed-form expressions for the generating function of the equilibrium queue size distribution. For the general case, we describe a computational procedure to obtain the distribution and the average of queue size in steady state. The numerical results obtained using this procedure are presented for a family of problems in which each message consists of two packets separated by a fixed time interval.*

## I. INTRODUCTION

A data communications network may be constructed by connecting together terminals and switching nodes so that each terminal is connected to just one node and the nodes are connected together in a more-or-less redundant fashion. All connections are by means of transmission lines. Those between terminals and nodes are called access lines, while those between one node and another are called trunk lines (Fig. 1). For economy, the access lines commonly have smaller bandwidth than the trunk lines.

The character of the traffic carried by a data network depends in part upon the type of terminal connected to it. Keyboard and display terminals transmit and receive data messages that are typically less than a few hundred characters in length. These terminals operate at speeds up to 1.2 Kb/s. Batch stations transmit and receive data in larger quantities and typically operate at speeds up to 9.6 Kb/s. Trunk lines, and the connections to computers, typically operate at about 50 Kb/s. Thus, we find that traffic in a data network is not uniformly distributed either among the terminals or among the various transmission lines.

Analysis of delay and the probability of queue overflow is most simply

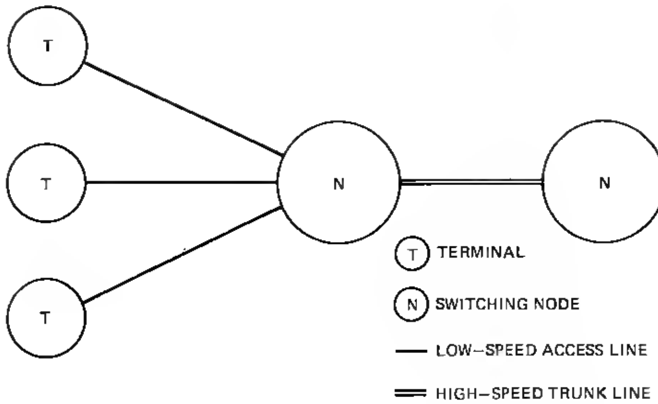


Fig. 1—Network topology.

obtained by assuming that all traffic is uniformly distributed between the terminals and that all transmission lines operate at the same speed. Using these assumptions, Chu[1,2] has studied two cases. In one case, equal size packets (individual characters perhaps) are generated randomly by the terminals. In another case, messages are generated at random, but each message consists of a random number of packets which all enter the network in one instant. The case of mixed input traffic was studied by Chu and Liang.[3]

Consider the situation when terminals randomly generate messages consisting of several fixed-size packets. The data are fed into a switching node over access lines that are substantially slower than the trunk lines used to carry data out of the node. Several packets may be transmitted on a trunk line in the time that it takes to transmit one packet on an access line (Fig. 2). Thus, packets of one message which are transmitted consecutively by a terminal will arrive periodically at the node, and the period will be greater than the period of packet transmissions on the

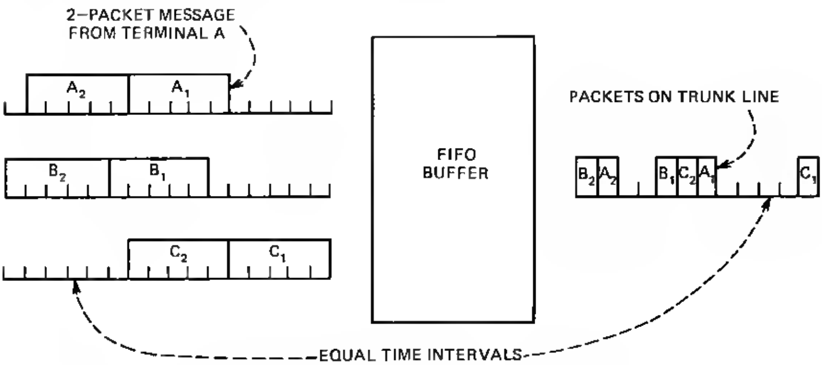


Fig. 2—Example of packet flow at a node.

trunk line. Given that packet arrivals are not entirely random, one would expect the buffer storage requirements in the switch to be less than is indicated by Chu and Liang's analyses.

In this paper, we study the behavior of a switching node tbat receives data from a (large) number of low-speed access lines. The data are received in the form of packets of a fixed size. As the packets arrive, they are placed in a buffer, which is a first-in-first-out queue. In an actual communications network, the buffer has a finite size, and a packet is lost if the buffer is full when it attempts to enter it. The buffer transmits packets at a uniform rate onto a high-speed trunk, provided the buffer is not empty. A crucial question is how large the buffer should be in order that the probability of packet loss should be less than $10^{-4}$, say. In the mathematical analysis of the single queue corresponding to this buffer, we consider a buffer of unlimited size so that no overflow is possible, and we calculate the steady-state probability that the buffer content (i.e., the number of packets in the buffer, or queue size) exceeds the proposed size of the finite buffer. We refer to this quantity as the probability of overflow, since it is usually used to estimate the actual overflow probability, when the probability of packet loss is very small.

Some of the symbols used are listed in Section II. The mathematical model, which includes assumptions concerning the packet arrivals, is discussed in Section III. This model was analyzed by two of the authors,[4] and formulas for calculating the equilibrium queue size distribution are summarized in Appendix A. These formulas involve some marginal distributions, and some polynomials which have to be determined. Explicit analytical expressions for the coefficients in these polynomials are given for some particular examples in Section IV. The computation of the coefficients for more general examples is discussed in Section V. The computation of the marginal distributions is discussed in Section VI. Some numerical results are presented in Section VII.

## II. NOTATION

| | |
|---|---|
| $b_n$ | buffer content at time $n$ |
| $z_n$ | number of packets entering buffer in time interval $(n,n+1]$ |
| $k+1$ | number of time intervals required for message arrival |
| $\alpha_j^i$ | nonnegative integers with $\alpha_0^i > 0$ |
| $(x_n^1, \ldots, x_n^l)$ | independent identically distributed vector of nonnegative integer valued random variables. |

## III. MATHEMATICAL MODEL

The behavior of a switching node is modeled by considering the state of the buffer at discrete times. Suppose that the buffer transmits one packet in a unit time interval. If $b_n$ denotes the buffer content at time $n$, the buffer content at time $n+1$ is

$$b_{n+1} = b_n - 1 + z_n \quad \text{if } b_n \geqslant 1$$
$$= z_n \quad \text{if } b_n = 0$$

or equivalently

$$b_{n+1} = (b_n - 1)^+ + z_n, \tag{1}$$

where $z_n$ is the number of packets entering the buffer in the time interval $(n,n+1]$. The peculiar feature of our queuing problem arises from the fact that the access lines are slower than the trunk line on which the output of the buffer is transmitted. For example, the 9.6 Kb/s access lines operate approximately at only one-fiftb of the speed of a 50 Kb/s trunk. Then, in our discrete model, messages arriving from the access lines will consist of a random number of packets separated by five units of time.

Let us first focus our attention on the case when there are two packets to each message. If the packets are separated by $d$ units of time, then the number of packets entering the buffer in the interval $(n,n+1]$ will be equal to the number of first packets in messages arriving in that interval, plus the number of second packets in messages whose first packets arrived $d$ intervals earlier. Then $z_n = x_n + x_{n-d}$, where $x_n$ is the number of first packets arriving in the interval $(n,n+1]$. Now consider the case in which the messages are transmitted by $N$ independent sources. The probability is zero that the first packet of a message from a source enters the buffer in a unit time interval if the first packet of a message from that source entered the buffer in one of the previous $(2d-1)$ time intervals, and otherwise it is $p$. Then

$$\Pr\{x_n = i_0 | x_{n-j} = i_j, \quad j = 1,2,\ldots\} = \binom{N-I}{i_0} p^{i_0}(1-p)^{N-I-i_0},$$

where $I = \Sigma_{j=1}^{2d-1} i_j$. Thus, $x_n$ and $x_{n-d}$ are not independent random variables. However, if $N \to \infty$ with $Np = \lambda$ fixed, then $\Pr\{x_n = i_0\} \to (e^{-\lambda}\lambda^{i_0})/i_0!$, independently of $x_{n-j}$, $j = 1,2,\ldots$. Hence, if the number of sources is large, it is reasonable to assume that the random variables $x_n$, $n = 0,1,\ldots$, are independently and identically distributed, and we will not restrict ourselves to the Poisson distribution.

Next we consider the case where each message consists of either two packets, separated by $d$ units of time, or of only one packet. If we let $x_n^1$ denote the number of first packets of two-packet messages arriving in the interval $(n,n+1]$, and $x_n^2$ denote the number of single packet messages arriving in the same interval, then $z_n = x_n^1 + x_{n-d}^1 + x_n^2$. In order to allow for randomness in the number of packets to a message (either one or two), $x_n^1$ and $x_n^2$ may be dependent on each other. For example, if the number of packets in a message is two with fixed probability $1-\rho$, and one with probability $\rho$, where $0 \leqslant \rho \leqslant 1$, then $E(t_1^{x_n{}^1} t_2^{x_n{}^2}) =$

$\Theta[(1-\rho)t_1+\rho t_2]$, where $\Theta(t)$ is the distribution for the number of messages arriving in a unit time interval. As before, when there is a large number of independent sources, it is reasonable to assume that $(x_n^1, x_n^2)$, $n = 0,1,\ldots$, are independent identically distributed vector random variables.

There are some obvious generalizations of the above special arrival processes. For instance, the random number of packets to a message may be as large as $m \geqslant 2$. Also, there may be slow access lines with several different speeds, so that the packets in a message may be separated by $d_i$ units of time, $i = 1,\ldots,r$. This led two of the authors[4] to consider arrival processes of the form

$$z_n = \sum_{i=1}^{l} \sum_{j=0}^{k} \alpha_j^i x_{n-j}^i, \tag{2}$$

where the vector nonnegative integer valued random variables $\{(x_n^1,\ldots,x_n^l)\}$ are independent and identically distributed, and $\alpha_j^i$ are nonnegative integers with $\alpha_0^i > 0$. In Appendix A we summarize the relevant results pertaining to tbe steady-state distribution of the queue size corresponding to (1) subject to (2).

## IV. EXPLICIT EXAMPLES

To calculate the generating function of the steady-state queue size from (37) and (38), it is necessary to know the polynomials $c_r(s)$, $r = 1,\ldots,k$, as well as the quantities $\phi_{rv}(s)$, $r = 0,\ldots,k$. Hence, from (39), we need to know the constants $c_j$. We now give explicit analytical results for some particular examples.

We first consider the arrival process

$$z_n = x_n^1 + x_{n-k}^1 + x_n^2 + x_n^3, \tag{3}$$

where $k \geqslant 2$, and

$$E(t_1^{x_n^1} t_2^{x_n^2} t_3^{x_n^3}) = \Theta[(1-\rho)t_1+\rho t_2]\Psi(t_3), \tag{4}$$

with $0 \leqslant \rho \leqslant 1$ fixed. This corresponds to arrivals from two different classes of sources. One class of sources sends messages which consist either of two packets separated by $k$ units of time with probability $1-\rho$, or of one packet with probability $\rho$. The other class of sources sends messages which consist of just one packet. From (2), (3), and (4), and the definition of $v_{rn}$ in (31), it follows that

$$\begin{aligned} \phi_{rv}(s) &= \Theta(s)\Psi(s), \quad r = 0,\ldots,k-1, \\ \phi_{kv}(s) &= \Theta[(1-\rho)s^2+\rho s]\Psi(s). \end{aligned} \tag{5}$$

It remains to give the values of $c_j$, wbich we do for $k = 2$ and 3, with $0 \leqslant \rho \leqslant 1$, and for $k = 4$ with $\rho = 0$ and $\Psi(s) \equiv 1$. It is shown in Appendix C how to determine which constants $c_j$ occur in (46). It is also shown how the values of $c_j$ were calculated in the case $k = 3$.

Let

$$\Theta(s) = \sum_{i=0}^{\infty} p_i s^i, \quad \Psi(s) = \sum_{i=0}^{\infty} q_i s^i. \tag{6}$$

For example, the case $p_i = e^{-\lambda}\lambda^i/i!$ corresponds to a Poisson distribution for the number of messages arriving from the first class of sources in a unit time interval. For $k = 2$ the nonzero coefficients $c_j$ are

$$c_{00}, \qquad c_{11} = (1-\rho)p_1 q_0 c_{00}, \tag{7}$$

with $c_{00} + c_{11} = 1$. For $k = 3$ the nonzero coefficients $c_j$ are

$$c_{000}, \qquad c_{011} = (1-\rho)p_1 q_0 c_{000},$$
$$c_{122} = (1-\rho)^2 p_1^2 q_0^2 c_{000}, \qquad c_{222} = (1-\rho)^2 p_0 p_2 q_0^2 c_{000}, \tag{8}$$

and

$$c_{111} = \frac{(1-\rho)q_0[p_1(1+p_0q_1) + 2\rho p_0 p_2 q_0]}{[1-(1-\rho)p_0 p_1 q_0^2]} c_{000}, \tag{9}$$

with $\Sigma c_j = 1$.

For $k = 4$ we take $\rho = 0$ and $\Psi(s) \equiv 1$, corresponding to $z_n = x_n + x_{n-4}$, with $E(s^{x_n}) = \Theta(s)$. In this case, tbere are 14 nonzero coefficients $c_j$, namely

$$c_{0000} = c_0, \qquad c_{0011} = p_1 c_0, \qquad c_{0122} = p_1^2 c_0,$$
$$c_{1233} = p_1^3 c_0, \qquad c_{0222} = p_0 p_2 c_0, \qquad c_{3333} = p_0^2 p_3 c_0,$$
$$c_{1333} = c_{2333} = c_{2233} = p_0 p_1 p_2 c_0, \tag{10}$$

and

$$c_{0111} = p_1 \Delta c_0, \qquad c_{1122} = p_1^2 \Delta c_0,$$
$$c_{1222} = p_1^2(1+p_0p_1)\Delta c_0, \qquad c_{2222} = p_0 p_2 (1+p_0p_1)\Delta c_0,$$
$$c_{1111} = p_1[1 + p_0^2(p_1^2 + p_0p_2)(1+p_0p_1)]\Delta c_0, \tag{11}$$

where

$$\Delta = \{1 - p_0 p_1[1 + p_0^2(p_1^2 + p_0 p_2)(1 + p_0 p_1)]\}^{-1}. \tag{12}$$

We now consider a class of arrival processes $z_n$ for which the polynomials $c_r(s)$, $r = 1, \ldots, k$, are, in fact, constants. Then the first two moments of the equilibrium queue size distribution, $Ey_0$ and $E(y_0^2)$, may be expressed, with the help of (59) to (61) and (63), in terms of the first three moments of $v_{rn}$, since $c'_r(1) \equiv 0$ and $c''_r(1) \equiv 0$. The class of arrival processes we consider corresponds to

$$\alpha_j^i > 0, \quad j = 0, \ldots, j_i, \quad i = 1, \ldots, l,$$
$$\alpha_j^i = 0, \textit{ otherwise}, \tag{13}$$

in (2). We will show that $y_{0n} = 0$ implies $y_{rn} = 0$, $r = 1, \ldots, k$, and hence that

$$\phi(0, s_1, \ldots, s_k) = \mu = 1 - Ev_{kn}. \tag{14}$$

This implies, from (39), that $c_r(s) = \mu$, $r = 1, \ldots, k$.

Now, since $b_n = y_{0n}$, it follows from (29) and (13) that $y_{0n} = 0$ implies that

$$x^i_{n-j-1} = 0, \quad j = 0, \ldots, j_i, \quad i = 1, \ldots, l. \tag{15}$$

If $r + 1 \geqslant j_i$, then $\alpha^i_j = 0$ for $j \geqslant r + 1$. If, on the other hand, $r + 1 \leqslant j_i$, then $x^i_{n-j+r} = 0$ for $r + 1 \leqslant j \leqslant j_i$, and $\alpha^i_j = 0$ for $j > j_i$. Hence, $\alpha^i_j x^i_{n-j+r} = 0$ for $j = r + 1, \ldots, k$, $i = 1, \ldots, l$, $r = 0, \ldots, k - 1$. Thus, from (30), $y_{0n} = 0$ implies $y_{rn} = 0$, $r = 1, \ldots, k$, as was asserted.

As a particular example, we consider messages which consist either of one packet or of two or more packets (up to a maximum number) which arrive in consecutive time intervals. Such is the case when the access lines have the same speed as the output line. For this example, the arrival process is

$$z_n = \sum_{i=1}^{k+1} \sum_{j=0}^{i-1} x^i_{n-j}. \tag{16}$$

Hence, from (2), $l = k + 1$ and

$$\alpha^i_j = \begin{matrix} 1, & j = 0, \ldots, i - 1, i = 1, \ldots, k + 1 \\ 0, & j = i, \ldots, k, i = 1, \ldots, k \end{matrix} \tag{17}$$

so that (13) is satisfied, and $c_r(s) = \mu$, $r = 1, \ldots, k$. From (31) it follows that

$$\mu^i_r = \min(i, r + 1), \quad r = 0, \ldots, k, \quad i = 1, \ldots, k + 1 \tag{18}$$

and

$$v_{0n} = \sum_{i=1}^{k+1} x^i_n,$$

$$v_{rn} = \sum_{i=1}^{r} i x^i_n + (r + 1) \sum_{i=r+1}^{k+1} x^i_n, \quad r = 1, \ldots, k. \tag{19}$$

Let $\rho_i \geqslant 0$, $i = 1, \ldots, k + 1$, be the probability that there are $i$ packets in a message, where $\Sigma_{i=1}^{k+1} \rho_i = 1$. Then,

$$E(t_1^{x^1_n} \ldots t_{k+1}^{x^{k+1}_n}) = \Theta\left(\sum_{i=1}^{k+1} \rho_i t_i\right). \tag{20}$$

Hence, from (19),

$$\phi_{0v}(s) = \Theta(s),$$

$$\phi_{rv}(s) = \Theta\left(\sum_{i=1}^{r} \rho_i s^i + \sum_{i=r+1}^{k+1} \rho_i s^{r+1}\right), \quad r = 1, \ldots, k. \tag{21}$$

## V. COMPUTATION OF $c_r(s)$

As described in Appendix A [see (34)], the constants $\{c_{\mathbf{j}}\}$ determine the polynomials $c_r(s)$ from (39). In principle, using a program like ALTRAN,[5] which performs symbolic manipulations, it is possible to substitute (34) into (33) and equate coefficients of like powers on both sides of (33) to get the equations for the $\{c_{\mathbf{j}}\}$. However, we will describe an alternate method that was used for obtaining the numerical results presented in Section VII. From (36) it is seen that

$$\mu c_{j_1 j_2 \ldots j_k} = P_{0 j_1 j_2 \ldots j_k}. \tag{22}$$

The method presented here arrives at equations for $\{P_{0\mathbf{j}}\}$ directly from the equations for $\{P_{\mathbf{i}}\}$: from (32)

$$P_{\mathbf{i}} = \sum_{j_0=0, \mathbf{j} \epsilon A} p_{i_0-j_1, i_1-j_2, \ldots i_k-j_k} P_{\mathbf{j}} + \sum_{j_0>0, \mathbf{j} \epsilon A} p_{i_0-j_1+1, i_1-j_2+1, \ldots, i_k-j_k+1} P_{\mathbf{j}}. \tag{23}$$

The sequence of programs that were used to determine $P_{0j_1 \ldots j_k}$ are described briefly below.

(*i*) The first step is to generate a sufficient number of equations from (23). Starting with $\mathbf{i} = (0,0, \ldots ,0)$, the right-hand side of (23) is found in symbolic form. Using a test for determining admissible states, every index $\mathbf{j}$ appearing on the right-hand side of (23) that corresponds to states not communicating with $(0,0, \ldots ,0)$ is omitted. Corresponding to each new index that arises on the right-hand side of (23), a new equation is generated from (23) by setting $\mathbf{i}$ equal to the new index. This process is terminated when every new index generated has $i_0 \geqslant k$. This whole process is repeated starting with each index with $i_0 = 0$, i.e., for each $P_{0j_1 \ldots j_k}$. The total number of equations and the total number of unknowns are counted. The output of this program is this set of equations in symbolic form. The number of equations turns out to be always less than the number of unknowns. However, we know[4] a set of linear homogeneous equations must exist for $P_{0j_1 \ldots j_k}$. In the examples considered, visual inspection revealed a few substitutions which made the number of equations one less than the number of unknowns. The details of a specific procedure for accomplishing this, in the case $l = 1$ in (2), have recently been given by Massey and Morrison.[6]

(*ii*) The normalizing constant which determines $P_{0j_1 \ldots j_k}$ uniquely is easily seen to be $(\mu/\Sigma P_{0j_1 \ldots j_k})$. This program starts with one of the unknowns found in step (*i*) above and sets its value equal to 1. Then it determines the maximum number of other unknowns that can be determined from this recursively, i.e., without having to invert any matrix. The best unknown to fix, the one that minimizes the number of unknowns to be solved by inverting a matrix, is selected and set equal

to 1. The matrix corresponding to the equations for the other unknowns, and the right-hand sides for these equations, are then generated in symbolic form. The output of this program is then a subroutine which generates the coefficients of $c_r(s)$ in symbolic form, as functions of the given probabilities. Some excerpts are shown in Tables I and II. In Table I is the "triangular" part of the equations for $k = 5$. $R_{i_0 \ldots i_5}$ here is $P_{i_0 \ldots i_5}$ normalized such that $P_{000000} = 1$ and $P(i) = \Pr\{x_n = i - 1\}$. In Table II are equations for generating the coefficients of $c_r(s)$ from $R_{i_0 \ldots i_5}$.

## VI. COMPUTATION OF MARGINAL DISTRIBUTIONS

Equations (37) and (38) give expressions for the generating functions of the equilibrium distributions of $y_{jn}$, $j = 0,1, \ldots ,k$. As in (30), $y_{0n}$ is $b_n$ the queue length. Hence, the equilibrium queue size distribution has as its generating function $\phi_0(s)$. If $\pi_{0j} = \lim_{n\uparrow\infty} \Pr\{b_n = j\}$, $j = 0,1, \ldots ,$ then $\phi_0(s) = \Sigma_{j=0}^{\infty} \pi_{0j} s^j$. One way to find $\{\pi_{0j}\}$ is to start with $\phi_k(s)$ and iterate using (38), thus obtaining an expression for $\phi_0(s)$, then to inverse transform $\phi_0(s)$. For example, using $s = e^{-j\omega}$, we can treat $\phi_0(s)$ as a function of $\omega$, and then finding $\{\pi_{0j}\}$ corresponds to finding the Fourier series for $\phi_0(e^{-j\omega})$.

We will present a different method here. Generally, the quantities of interest are

$$\Pi_{0j} = \sum_{i=0}^{j} \pi_{0i} = \lim_{n\uparrow\infty} \Pr\{b_n \leqslant j\} \quad \text{for } j \leqslant N,$$

Table I

| | |
|---|---|
| $R_{111111} = ((1-P(1))/P(1))^*R_{000000}$ | $R_{222356} = (P(2)/P(1))^*R_{111244}$ |
| $R_{111112} = (P(2)/P(1))^*R_{000000}$ | $R_{222446} = (P(3)/P(2))^*R_{111334}$ |
| $R_{222223} = (P(2)/P(1))^*R_{111111}$ | $R_{222456} = (P(2)/P(1))^*R_{111344}$ |
| $R_{222224} = (P(3)/P(2))^*R_{111112}$ | $R_{222556} = (P(2)/P(1))^*R_{111444}$ |
| $R_{333335} = (P(3)/P(2))^*R_{222223}$ | $R_{222366} = P(1)^*R_{333347}$ |
| $R_{333336} = (P(4)/P(3))^*R_{222224}$ | $R_{222466} = P(1)^*R_{333357}$ |
| $R_{444447} = (P(4)/P(3))^*R_{333335}$ | $R_{222566} = P(1)^*R_{333367}$ |
| $R_{444448} = (P(5)/P(4))^*R_{333336}$ | $R_{222666} = P(1)^*R_{333377}$ |
| $R_{000011} = P(1)^*R_{111112}$ | $R_{001233} = P(1)^*R_{111234}$ |
| $R_{111123} = (P(2)/P(1))^*R_{000011}$ | $R_{001333} = P(1)^*R_{111244}$ |
| $R_{111133} = P(1)^*R_{222224}$ | $R_{002233} = P(1)^*R_{111334}$ |
| $R_{222235} = (P(3)/P(2))^*R_{111123}$ | $R_{002333} = P(1)^*R_{111344}$ |
| $R_{222245} = (P(2)/P(1))^*R_{111133}$ | $R_{003333} = P(1)^*R_{111444}$ |
| $R_{222255} = P(1)^*R_{333336}$ | $R_{112345} = (P(2)/P(1))^*R_{001233}$ |
| $R_{333347} = (P(4)/P(3))^*R_{222235}$ | $R_{112445} = (P(2)/P(1))^*R_{001333}$ |
| $R_{333357} = (P(3)/P(2))^*R_{222245}$ | $R_{113345} = (P(2)/P(1))^*R_{002233}$ |
| $R_{333367} = (P(2)/P(1))^*R_{222255}$ | $R_{113445} = (P(2)/P(1))^*R_{002333}$ |
| $R_{333377} = P(1)^*R_{444448}$ | $R_{114445} = (P(2)/P(1))^*R_{003333}$ |
| $R_{000122} = P(1)^*R_{111123}$ | $R_{112355} = P(1)^*R_{222346}$ |
| $R_{000222} = P(1)^*R_{111133}$ | $R_{112455} = P(1)^*R_{222356}$ |
| $R_{111234} = (P(2)/P(1))^*R_{000122}$ | $R_{112555} = P(1)^*R_{222366}$ |
| $R_{111334} = (P(2)/P(1))^*R_{000222}$ | $R_{113355} = P(1)^*R_{222446}$ |
| $R_{111244} = P(1)^*R_{222235}$ | $R_{113455} = P(1)^*R_{222456}$ |
| $R_{111344} = P(1)^*R_{222245}$ | $R_{113555} = P(1)^*R_{222466}$ |
| $R_{111444} = P(1)^*R_{222255}$ | $R_{114455} = P(1)^*R_{222556}$ |
| $R_{222346} = (P(3)/P(2))^*R_{111234}$ | $R_{114555} = P(1)^*R_{222566}$ |

Table II

$C(1,1) = R_{000000} + R_{000011} + R_{000111} + R_{000122} + R_{000222} + R_{001111} + R_{001122}$
$C(1,1) = C(1,1) + R_{001222} + R_{001233} + R_{001333} + R_{002222} + R_{002233} + R_{002333} + R_{003333}$
$C(1,2) = R_{011111} + R_{011122} + R_{011222} + R_{011233} + R_{011333} + R_{012222} + R_{012233}$
$C(1,2) = C(1,2) + R_{012333} + R_{012344} + R_{012444} + R_{013333} + R_{013344} + R_{013444} + R_{014444}$
$C(1,3) = R_{022222} + R_{022233} + R_{022333} + R_{022344} + R_{022444} + R_{023333} + R_{023344}$
$C(1,3) = C(1,3) + R_{023444} + R_{024444}$
$C(1,4) = R_{033333} + R_{033344} + R_{033444} + R_{034444}$
$C(1,5) = R_{044444}$
$C(2,1) = R_{000000} + R_{000011} + R_{000111} + R_{000122} + R_{000222}$
$C(2,2) = R_{001111} + R_{001122} + R_{001222} + R_{001233} + R_{001333} + R_{011111} + R_{011122}$
$C(2,2) = C(2,2) + R_{011222} + R_{011233} + R_{011333}$
$C(2,3) = R_{002222} + R_{002233} + R_{002333} + R_{012222} + R_{012233} + R_{012333} + R_{012344}$
$C(2,3) = C(2,3) + R_{012444} + R_{022222} + R_{022233} + R_{022333} + R_{022344} + R_{022444}$
$C(2,4) = R_{003333} + R_{013333} + R_{013344} + R_{013444} + R_{023333} + R_{023344} + R_{023444}$
$C(2,4) = C(2,4) + R_{033333} + R_{033344} + R_{033444}$
$C(2,5) = R_{014444} + R_{024444} + R_{034444} + R_{044444}$
$C(3,1) = R_{000000} + R_{000011}$
$C(3,2) = R_{000111} + R_{000122} + R_{0001111} + R_{001122} + R_{011111} + R_{011122}$
$C(3,3) = R_{000222} + R_{001222} + R_{001233} + R_{002222} + R_{002233} + R_{011222} + R_{011233}$
$C(3,3) = C(3,3) + R_{012222} + R_{012233} + R_{022222} + R_{022233}$
$C(3,4) = R_{001333} + R_{002333} + R_{003333} + R_{011333} + R_{012333} + R_{012344} + R_{013333}$
$C(3,4) = C(3,4) + R_{013344} + R_{022333} + R_{022344} + R_{023333} + R_{023344} + R_{033333} + R_{033344}$
$C(3,5) = R_{012444} + R_{013444} + R_{014444} + R_{022444} + R_{023444} + R_{024444} + R_{033444}$
$C(3,5) = C(3,5) + R_{034444} + R_{044444}$
$C(4,1) = R_{000000}$
$C(4,2) = R_{000011} + R_{000111} + R_{001111} + R_{011111}$
$C(4,3) = R_{000122} + R_{000222} + R_{001122} + R_{001222} + R_{002222} + R_{011122} + R_{011222}$
$C(4,3) = C(4,3) + R_{012222} + R_{022222}$
$C(4,4) = R_{001233} + R_{001333} + R_{002233} + R_{002333} + R_{003333} + R_{011233} + R_{011333}$
$C(4,4) = C(4,4) + R_{012233} + R_{012333} + R_{013333} + R_{022233} + R_{022333} + R_{023333} + R_{033333}$
$C(4,5) = R_{012344} + R_{012444} + R_{013344} + R_{013444} + R_{014444} + R_{022344} + R_{022444}$
$C(4,5) = C(4,5) + R_{023344} + R_{023444} + R_{024444} + R_{033344} + R_{033444} + R_{034444} + R_{044444}$

where $N$ is some initially selected constant. The method presented here determines $\Pi_{0j}$, $j \leqslant N$, explicitly in a finite number of additions and multiplications. Let $\phi_r(s) = \Sigma_{j=0}^{\infty} \pi_{rj} s^j$ and $\Pi_{rj} = \Sigma_{i=0}^{j} \pi_{ri}$. We can write (37) as

$$\phi_k(s) = \frac{(1-s)\phi_{kv}(s) \cdot c_k(s)}{\phi_{kv}(s) - s}. \tag{24}$$

Denote $\phi_k(s)/(1-s)$ by $\psi_k(s)$, so that $\psi_k(s) = \Sigma_{j=0}^{\infty} \Pi_{kj} s^j$ for $|s| < 1$. Therefore,

$$\psi_k(s) = \frac{c_k(s) \cdot \phi_{kv}(s)}{\phi_{kv}(s) - s}, \quad |s| < 1. \tag{25}$$

Similarly defining $\psi_r(s) = \Sigma_{j=0}^{\infty} \Pi_{rj} s^j$ for $r = k-1, k-2, \ldots, 0$, we have

$$\psi_r(s) = s^{-1}[\psi_{r+1}(s) - c_{r+1}(s)]\phi_{rv}(s). \tag{26}$$

The functions $\phi_{rv}(s)$ are, of course, determined from the distributions of $x_n^i$ and the constants $\alpha_j^i$.

For each $r$, the process of determining $\Pi_{rj}$ from $\Pi_{r+1,j}$ can be described as follows. Subtract the known constants $c_{r+1,j}$ from $\Pi_{r+1,j}$ for $j \leqslant$ degree

of $c_{r+1}(s)$. Since $\Pi_{r+1,0} = c_{r+1,0}$, the sequence $\delta_{rj}$ corresponding to $s^{-1}[\psi_{r+1}(s) - c_{r+1}(s)]$ is such that $\delta_{rj} = 0$ for $j < 0$ and

$$\delta_{rj} = \begin{array}{l} \Pi_{r+1,j+1} - c_{r+1,j+1} \text{ for } 1 \leqslant j+1 \leqslant \textit{degree of } c_{r+1}(s) \\ \Pi_{r+1,j+1} \text{ for } j+1 > \textit{degree of } c_{r+1}(s) \end{array}$$

In order to get $\{\Pi_{rj}\}$, we now convolve this sequence $\{\delta_{rj}\}$ with the sequence corresponding to $\phi_{rv}(s)$, say $\{p_{rj}\}$. Therefore, $\Pi_{rj} = \Sigma_{i=0}^{j} \delta_{r,j-i} p_{ri}$. This process involves only a finite number of multiplications and additions as long as only a finite number of $\Pi_{rj}$'s are sought. Notice that in order to determine $\Pi_{0j}$, $j \leqslant N$, we have to start with values of $\Pi_{kj}$ for $j \leqslant N + k$.

We will now show that there is a recursion which allows us to compute $\Pi_{kj,}$ $j \leqslant N + k$, in a finite number of arithmetic operations. Let $\Pi'_{kj}$ correspond to $\phi_{kv}(s)/(\phi_{kv}(s) - s)$ and let $\phi_{kv}(s) = \Sigma_{j=0}^{\infty} p_{kj}s^j$. Then, equating coefficients of like powers of $s$ on both sides of

$$\sum_{j=0}^{\infty} \Pi'_{kj}s^j = \frac{\phi_{kv}(s)}{\phi_{kv}(s) - s}, \tag{27}$$

we can derive the following.

$$\Pi'_{k0} = 1,$$

$$\Pi'_{kj} = (p_{kj} + \Pi'_{k,j-1} - \sum_{i=1}^{j} p_{ki}\Pi'_{k,j-i})/p_{k0}, \quad j = 1,2,\ldots. \tag{28}$$

Once the $\Pi'_{kj}$ have been determined, we convolve the sequence $\{\Pi'_{kj}\}$ with $\{c_{kj}\}$ to get $\{\Pi_{kj}\}$, as seen from (25). Summarizing, we have shown that $\Pi_{0j}$ for $j \leqslant N$ can be determined from $c_r(s)$, $r = 1, \ldots, k$ and $\phi_{rv}(s)$, $r = 0,1,\ldots,k$, by performing only a finite number of multiplications and additions. The method described was used in calculating the probabilities presented in Section VII.

## VII. AN EXAMPLE OF THE CALCULATIONS

We calculated the queue size distributions for various traffic intensities $Ez_n$ for the following queuing models:

(*i*) $b_{n+1} = (b_n - 1)^+ + 2x_n, \quad k = 0$
(*ii*) $b_{n+1} = (b_n - 1)^+ + x_n + x_{n-3}, \quad k = 3$
(*iii*) $b_{n+1} = (b_n - 1)^+ + x_n + x_{n-4}, \quad k = 4$
(*iv*) $b_{n+1} = (b_n - 1)^+ + x_n + x_{n-5}, \quad k = 5,$

where the i.i.d. random variables $x_n$ are assumed to be distributed according to the Poisson law. These cases are referred to equivalently by referring to the value of $k$. The objective was to determine what buffer size would suffice for a concentrator used to buffer terminals that generate two packets per message but are slower than the trunk line. We

present here some of the results. In Fig. 3 the abscissa corresponds to traffic intensity: average number of packets transmitted on the trunk line per unit of time. In the steady state, the probability that the queue size exceeds that shown on the ordinate is less than $10^{-4}$ for each value of traffic intensity. The cases when all packets arrive at once ($k = 0$) or "infinitely apart" ($k \uparrow \infty$) are shown, as well as the case $k = 5$. When $k = 0$, the queue size corresponding to each value of traffic intensity is either twice that of the case $k = \infty$, or one less than twice.[4] These two cases are further compared in Fig. 4, this time for buffer sizes 20 and 40, and the logarithm to base 10 of the probability of the queue size exceeding 20 and 40 is plotted.

For fixed traffic intensities, the change in the probability of overflow as a function of buffer size can be seen from Figs. 5, 6, and 7. From these figures we can see that for low traffic intensities and low buffer sizes there is a difference between the batch case $k = 0$, and $k = 5$ (see Fig. 5), but for larger traffic intensities the difference decreases substantially. From the formulas (37) and (38), it can be shown that the tail of queue size distribution is geometric for each value of $k$. Furthermore, for all finite values of $k$, the common ratio is the same as that of the case when $k = 0$, so that the similarity in the behavior of queue size distributions for large values of queue size is as expected. The slopes of the curves marked $k = 5$ in Figs. 5, 6, and 7 approach those of the curves marked $k = 0$ for large values of the abscissa. Hence, in applications where probabilities

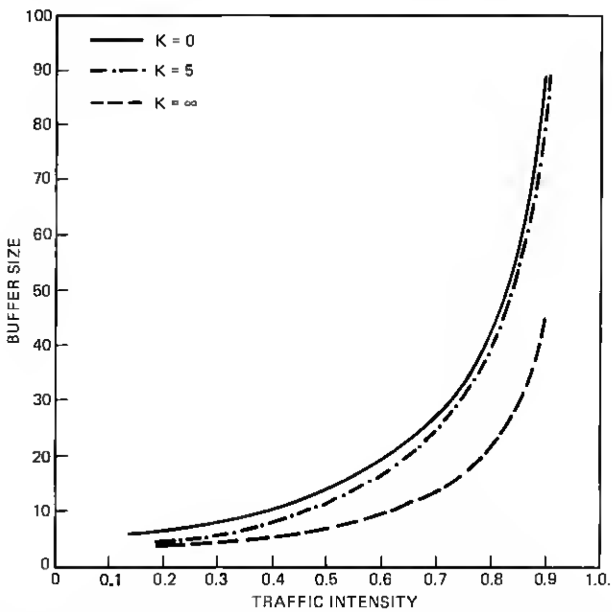


Fig. 3—Buffer size vs traffic intensity for $k = 0, 5$ and $\infty$, and probability of overflow $< 10^{-4}$.

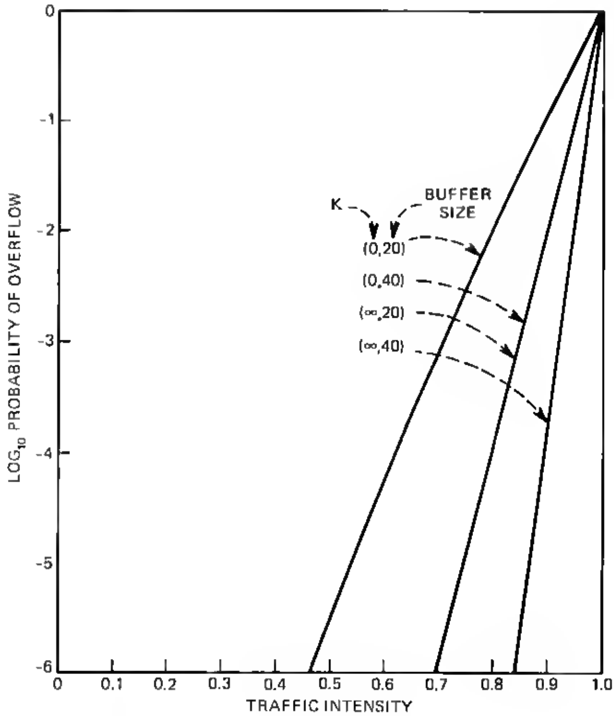


Fig. 4—Probability of overflow vs traffic intensity for $k = 0$ and $\infty$, and buffer sizes 20 and 40.

of overflow of $10^{-7}$ or smaller are required, the relative slowness of the sources of packets does not seem to reduce the buffer size required. Packets may be assumed to arrive simultaneously for the purpose of estimating the buffer size.

## APPENDIX A

### *Summary of Formulas*

We here summarize formulas for calculating the equilibrium queue size distribution. With a model for the input process $z_n$ as in (2), the queue size $b_n$ is described by

$$b_{n+1} = (b_n - 1)^+ + \sum_{i=1}^{l} \sum_{j=0}^{k} \alpha_j^i x_{n-j}^i. \tag{29}$$

Various formulas pertaining to (29) were derived.[4] The reader is referred to Ref. 4 for proofs of the formulas presented here. Define $y_{0n} = b_n$ and, for $r = 0,1, \ldots, k-1$,

$$y_{r+1,n} = y_{rn} + \sum_{i=1}^{l} \sum_{j=r+1}^{k} \alpha_j^i x_{n-j+r}^i. \tag{30}$$

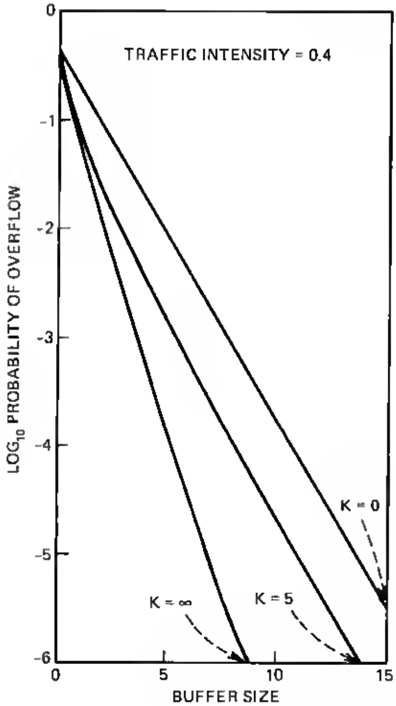


Fig. 5—Probability of overflow vs buffer size for $k = 0$, 5 and $\infty$, and traffic intensity 0.4.

Note that $y_{0n} \leqslant y_{1n} \leqslant \dots \leqslant y_{kn}$. The vector process $(y_{0n}, y_{1n}, \dots, y_{kn})$, $n = 0,1,2,\dots$, is Markov under the assumption that $(x_n^1, x_n^2, \dots, x_n^l)$, $n = 0,1,2,\dots$, is a vector sequence of independent identically distributed random variables. The Markov chain, denoted by $S$, which corresponds to $(y_{0n}, y_{1n}, \dots, y_{kn})$, was shown to be positive recurrent when $Ez_n < 1$. Tbe states of $S$ are those which communicate with $(0,0,\dots,0)$, since it is assumed that the buffer is empty and no one is transmitting initially, at $n = 0$. Let

$$\sum_{j=0}^{r} \alpha_j^i = \mu_r^i, \sum_{i=1}^{l} \mu_r^i x_n^i = v_{rn}, \quad r = 0,1,\dots,k. \tag{31}$$

The probabilities that enter the calculations turn out to be only those corresponding to the random variables $v_{rn}$, $r = 0,1,\dots,k$. Let $p_{i_0,i_1,\dots,i_k} = \Pr\{v_{0n} = i_0, v_{1n} = i_1, \dots, v_{kn} = i_k\}$. Then the transition probabilities for $S$ are given by:

$$P_{\mathbf{i}}^{n+1} = \sum_{j_0=0,\mathbf{j}\epsilon A} p_{i_0-j_1,i_1-j_2,\dots,i_k-j_k} P_{\mathbf{j}}^n + \sum_{j_0>0,\mathbf{j}\epsilon A} p_{i_0-j_1+1,i_1-j_2+1,\dots,i_k-j_k+1} P_{\mathbf{j}}^n. \tag{32}$$

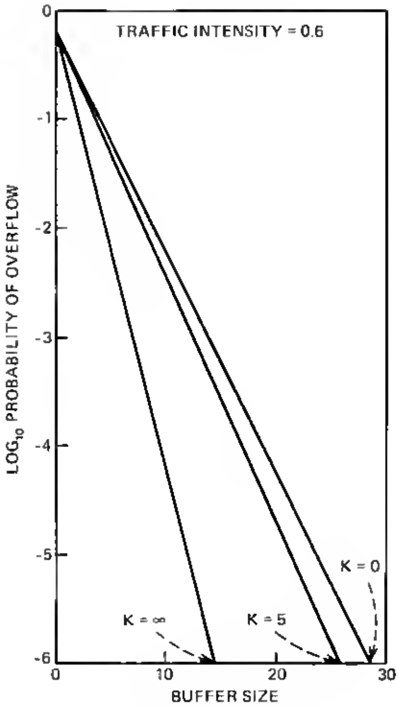


Fig. 6—Probability of overflow vs buffer size for $k = 0$, 5 and $\infty$, and traffic intensity 0.6.

The sums in (32) are over those indices $j_0, j_1, \ldots, j_k$ which correspond to states communicating with $(0,0,\ldots,0)$ denoted by $A$. As mentioned above, when $Ez_n < 1$, then $\lim_{n\uparrow\infty} P_{\mathbf{i}}^n = P_{\mathbf{i}}$ exists and $P_{\mathbf{i}}$ satisfies (32) with $P_{\mathbf{i}}^n$ and $P_{\mathbf{i}}^{n+1}$ both replaced by $P_{\mathbf{i}}$. The generating function corresponding to $P_{\mathbf{i}}$,

$$\phi(s_0, s_1, \ldots, s_k) = \sum_{\mathbf{i} \epsilon A} P_{\mathbf{i}} s_0^{i_0} s_1^{i_1} \ldots s_k^{i_k}$$

satisfies

$$\phi(s_0, s_1, \ldots, s_k) = \left[\phi(1, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k) \prod_{i=0}^{k} s_i{}^{-1} + \left(1 - \prod_{i=0}^{k} s_i{}^{-1}\right) \phi(0, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k)\right] \phi_v(s_0, s_1, \ldots, s_k), \quad (33)$$

where

$$\phi_v(s_0, s_1, \ldots, s_k) = E \prod_{i=0}^{k} s_i^{\nu_{in}}.$$

It can be shown[4] that $\phi$ has the representation

$$\phi(s_0, s_1, \ldots, s_k) = \sum_{\mathbf{j}} c_{\mathbf{j}} \theta_{\mathbf{j}}(s_0, \ldots, s_k), \quad (34)$$

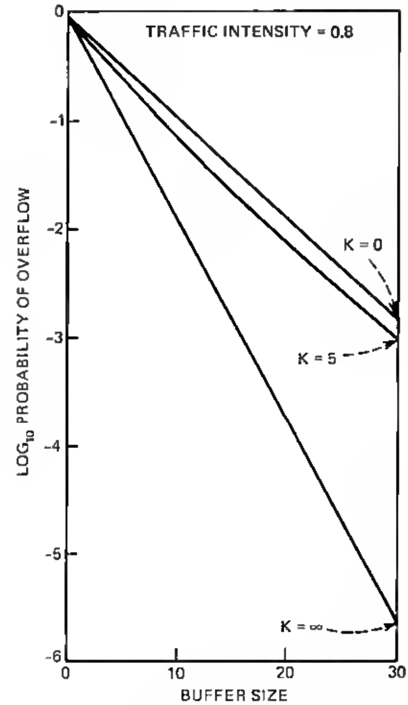


Fig. 7—Probability of overflow vs buffer size for $k = 0$, 5 and $\infty$, and traffic intensity 0.8.

where $\theta_j(s_0, \ldots, s_k)$ are solutions of

$$\theta_j(s_0,s_1, \ldots ,s_k) = \Big[\theta_j(1,s_0,s_1, \ldots ,s_{k-2},s_{k-1}s_k) \prod_{i=0}^{k} s_i^{-1}$$

$$+ \mu\Big(1 - \prod_{i=0}^{k} s_i^{-1}\Big)s_0^{j_1}s_1^{j_2} \ldots s_{k-2}^{j_{k-1}}(s_{k-1}s_k)^{j_k}\Big]\phi_v(s_0,s_1, \ldots ,s_k), \quad (35)$$

with $\mu = 1 - Ez_n = 1 - Ev_{kn}$. Substitution of (34) into (33) yields a system of equations for $c_j$ which together with $\Sigma c_j = 1$ uniquely determine

$$\phi(0,s_1, \ldots ,s_k) = \mu \sum_j c_j s_1^{j_1} \ldots s_k^{j_k}. \quad (36)$$

Once $\phi(0,s_1, \ldots ,s_k)$ is found, the generating functions corresponding to the marginal distributions, namely $\lim_{n\uparrow\infty} Es^{y_{in}} = \phi_i(s)$, are solutions of

$$\phi_k(s) = \frac{(1 - s^{-1})c_k(s)\phi_{kv}(s)}{1 - s^{-1}\phi_{kv}(s)}, \quad (37)$$

where $\phi_{jv}(s) = Es^{v_{jn}}$, $j = 0,1, \ldots ,k$, and, for $r = k - 1, \ldots ,1,0$,

$$\phi_r(s) = [s^{-1}\phi_{r+1}(s) + (1 - s^{-1})c_{r+1}(s)]\phi_{rv}(s). \quad (38)$$

Here the polynomials $c_r(s)$, $r = 1, \ldots, k$ are of finite degree and are given by

$$c_r(s) = \phi(0,1, \ldots, s, 1, \ldots, 1) = \mu \sum_{\mathbf{j}}^{r} c_{j_1 j_2 \ldots j_r \ldots j_k} s^{j_r}. \tag{39}$$

Then (37) uniquely determines $\phi_k(s)$. Using (38) $k$ times yields $\phi_0(s)$, which by definition of $y_{0n}$ [see (30)] is the generating function of steady-state queue size.

An alternate representation of the generating function corresponding to $P_{\mathbf{i}}$ is obtained by setting $u_j = \Pi_{i=j}^{k} s_i$, $j = 0, \ldots, k$ and defining

$$\phi(s_0, s_1, \ldots, s_k) = \Phi(u_0, u_1, \ldots, u_k). \tag{40}$$

Then, corresponding to (33),

$$\begin{aligned}\Phi(u_0, u_1, \ldots, u_k) = {} & [u_0^{-1}\Phi(u_0, u_0, u_1, \ldots, u_{k-1}) \\ & + (1 - u_0^{-1})\Phi(0, u_0, u_1, \ldots, u_{k-1})]\Phi_v(u_0, u_1, \ldots, u_k),\end{aligned} \tag{41}$$

where

$$\Phi_v(u_0, u_1, \ldots, u_k) = E\left(\prod_{r=0}^{k} u_r^{w_{rn}}\right), \quad w_{rn} = \sum_{i=1}^{l} \alpha_r^i x_n^i. \tag{42}$$

It follows from (41) that, for $j = 0, 1, \ldots, k-2$, $(k \geq 2)$,

$$\begin{aligned}\Phi(s, \ldots, s, u_1, \ldots, u_{k-j}) = {} & [s^{-1}\Phi(s, \ldots, s, u_1, \ldots, u_{k-j-1}) \\ & + (1 - s^{-1})\Phi(0, s, \ldots, s, u_1, \ldots, u_{k-j-1})] \\ & \times \Phi_v(s, \ldots, s, u_1, \ldots, u_{k-j}),\end{aligned} \tag{43}$$

and

$$\begin{aligned}\Phi(s, \ldots, s, u_1) = {} & [s^{-1}\Phi(s, \ldots, s) \\ & + (1 - s^{-1})\Phi(0, s, \ldots, s)]\Phi_v(s, \ldots, s, u_1).\end{aligned} \tag{44}$$

If we set $u_1 = s$ in (44), and solve for $\Phi(s, \ldots, s)$, we obtain

$$\Phi(s, \ldots, s) = \frac{(1-s)\Phi(0, s, \ldots, s)\Phi_v(s, \ldots, s)}{[\Phi_v(s, \ldots, s) - s]}. \tag{45}$$

It was shown[4] that $\Phi(0, u_1, \ldots, u_k)$ is a multinomial independent of $u_k$. From (43) to (45), $\Phi(s, u_1, \ldots, u_k)$ may be expressed in terms of $\Phi(0, s, \ldots, s, u_1, \ldots, u_{k-j-1})$, $j = 0, \ldots, k-2$, and $\Phi(0, s, \ldots, s)$. If we let $s \to 0$ in this expression, and equate $\Phi(0, u_1, \ldots, u_k)$ with the finite part, we obtain a system of homogeneous linear equations for the coefficients in the multinomial. In general, we also obtain a (consistent) set of homogeneous linear equations from finiteness conditions. The normalization condition is $\Phi(0, 1, \ldots, 1) = \mu$. From (36) and (40), it follows that

$$\Phi(0, u_1, \ldots, u_k) = \mu \sum_{\mathbf{j}} c_{j_1 \ldots j_k} u_1^{j_1} \prod_{i=2}^{k} u_i^{j_i - j_{i-1}}. \tag{46}$$

Also, from (39),

$$c_r(s) = \Phi(0,\overbrace{s,\ldots,s}^{r},\overbrace{1,\ldots,1}^{k-r}). \tag{47}$$

Formulas for calculating the first two moments of the equilibrium queue size distribution are derived in Appendix B.

## APPENDIX B

### *First and Second Moments*

We here derive expressions for the first two moments of $y_r = \lim_{n\to\infty} y_{rn}$. We note that $y_0$ represents the equilibrium queue size. By definition,

$$\phi_r(s) = E(s^{y_r}), \quad \phi_{rv}(s) = E(s^{v_{rn}}), \tag{48}$$

for $r = 0, \ldots, k$. Hence,

$$\phi_r(1) = 1, \quad \phi'_r(1) = Ey_r, \quad \phi''_r(1) = E(y_r^2) - Ey_r \tag{49}$$

and

$$\phi_{rv}(1) = 1, \quad \phi'_{rv}(1) = Ev_{rn}, \quad \phi''_{rv}(1) = E(v_{rn}^2) - Ev_{rn}. \tag{50}$$

We also note that, from (39), since $\Sigma\, c_j = 1$,

$$c_r(1) = \mu = 1 - Ev_{kn}, \quad r = 1, \ldots, k. \tag{51}$$

From (37),

$$\phi_k(s) = c_k(s)\phi_{vk}(s)f(s), \quad f(s) = \frac{(1-s)}{[\phi_{kv}(s) - s]}, \tag{52}$$

and, from (38),

$$\phi_r(s) = [s^{-1}\phi_{r+1}(s) + (1-s^{-1})c_{r+1}(s)]\phi_{rv}(s), \quad r = 0, \ldots, k-1. \tag{53}$$

If we differentiate (53), we obtain

$$\begin{aligned}\phi'_r(s) = {} & [-s^{-2}\phi_{r+1}(s) + s^{-1}\phi'_{r+1}(s) + s^{-2}c_{r+1}(s) + (1-s^{-1})c'_{r+1}(s)] \\ & \times \phi_{rv}(s) + [s^{-1}\phi_{r+1}(s) + (1-s^{-1})c_{r+1}(s)]\phi'_{rv}(s).\end{aligned} \tag{54}$$

If we set $s = 1$ in (54), and use (49) to (51), we obtain

$$Ey_r = Ey_{r+1} + Ev_{rn} - Ev_{kn}, \quad r = 0, \ldots, k-1. \tag{55}$$

Next, if we differentiate (52), we find that

$$\phi'_k(s) = [c'_k(s)\phi_{kv}(s) + c_k(s)\phi'_{kv}(s)]f(s) + c_k(s)\phi_{kv}(s)f'(s). \tag{56}$$

But

$$\phi_{kv}(s) = 1 + (s-1)\phi'_{kv}(1) + \frac{1}{2}(s-1)^2\phi''_{kv}(s) + \frac{1}{6}(s-1)^3\phi'''_{kv}(1) + \ldots. \quad (57)$$

Hence, from (52),

$$f(s) = \frac{1}{[1-\phi'_{kv}(1)]} + \frac{(s-1)\phi''_{kv}(1)}{2[1-\phi'_{kv}(1)]^2} + \frac{(s-1)^2}{2}\left\{\frac{\phi'''_{kv}(1)}{3[1-\phi'_{kv}(1)]^2} + \frac{[\phi''_{kv}(1)]^2}{2[1-\phi'_{kv}(1)]^3}\right\} + \ldots. \quad (58)$$

If we let $s \to 1$ in (56), and use (49) to (51) and (58), we obtain

$$Ey_k = \frac{c'_k(1)}{(1-Ev_{kn})} + Ev_{kn} + \frac{[E(v_{kn}^2) - Ev_{kn}]}{2(1-Ev_{kn})}. \quad (59)$$

From (55), it follows that

$$Ey_j = Ey_k + \sum_{r=j}^{k-1} Ev_{rn} - (k-j)Ev_{kn}, \quad j = 0, \ldots, k-1. \quad (60)$$

This determines $Ey_j$, and in particular $Ey_0$, in view of (59).

For the second-order moments, if we differentiate (54) and set $s = 1$, we obtain

$$E(y_r^2) = E(y_{r+1}^2) - Ey_{r+1} + Ey_r + E(v_{rn}^2) - Ev_{rn} + 2(1-Ev_{rn})(Ev_{kn} - Ey_{r+1}) + 2c'_{r+1}(1), \quad (61)$$

for $r = 0, \ldots, k-1$. Finally, if we differentiate (56) and let $s \to 1$, and use the relationship

$$\phi'''_{kv}(1) = E(v_{kn}^3) - 3E(v_{kn}^2) + 2E(v_{kn}), \quad (62)$$

which follows from (48), we obtain, after some simplification,

$$E(y_k^2) = Ey_k + \frac{[c''_k(1) + 2(Ev_{kn})c'_k(1)]}{(1-Ev_{kn})} + \frac{[E(v_{kn}^2) - Ev_{kn}]c'_k(1)}{(1-Ev_{kn})^2} + \frac{[E(v_{kn}^3) - Ev_{kn}]}{3(1-Ev_{kn})} + \frac{[E(v_{kn}^2) - Ev_{kn}]^2}{2(1-Ev_{kn})^2}. \quad (63)$$

An expression for $E(y_0^2)$ may be obtained from $k$ applications of (61), with the help of (59), (60), and (63).

## APPENDIX C

### *Determination of Constants*

We first show how to determine which constants $c_j$ occur in (46) for the example of (3). From (2) and (30),

$$y_{0n} = b_n,\; y_{r+1,n} - y_{rn} = x^1_{n-k+r},\; r = 0, \ldots, k-1. \quad (64)$$

Hence,

$$\Phi(u_0, u_1, \ldots, u_k) = \lim_{n\to\infty} E\left(u_0^{b_n} \prod_{r=1}^{k} u_r^{x^1_{n-k+r-1}}\right). \quad (65)$$

But from iteration of (29), it is evident that

$$b_n = 0 \Rightarrow \sum_{j=1}^{l} z_{n-j} \leqslant l-1, \quad l = 1, \ldots, k, \tag{66}$$

where $z_n$ is given by (2). Hence, for the example of (3),

$$b_n = 0 \Rightarrow \sum_{j=1}^{l} x_{n-j}^1 \leqslant l-1, \quad l = 1, \ldots, k. \tag{67}$$

The inequalities in (67) determine the admissible vectors $(x_{n-k}^1, \ldots, x_{n-1}^1)$ corresponding to $b_n = 0$, and thus, from (65), which constants $c_j$ occur in (46). Note, in particular, that $x_{n-1}^1 = 0$ implies that $\Phi(0, u_1, \ldots, u_k)$ is independent of $u_k$.

For purposes of illustration, we show how to calculate the values of $c_j$ for the example of (3), subject to (4), in the case $k = 3$. From the above procedure it is found that

$$\Phi(0,u_1,u_2,u_3) = \chi(u_1,u_2) = \mu(c_{000} + c_{111}u_1 + c_{011}u_2 + c_{222}u_1^2 + c_{122}u_1u_2). \tag{68}$$

But from (2) and (42),

$$\Phi_v(u_0,u_1,u_2,u_3) = \Theta[(1-\rho)u_0u_3 + \rho u_0]\Psi(u_0). \tag{69}$$

Hence, from (45),

$$\Phi(s,s,s,s) = \frac{(1-s)\chi(s,s)\Theta[(1-\rho)s^2 + \rho s]\Psi(s)}{\{\Theta[(1-\rho)s^2 + \rho s]\Psi(s) - s\}}. \tag{70}$$

Then, from (44), we obtain

$$\Phi(s,s,s,u_1) = \frac{(1-s)\chi(s,s)\Theta[(1-\rho)su_1 + \rho s]\Psi(s)}{\{\Theta[(1-\rho)s^2 + \rho s]\Psi(s) - s\}}. \tag{71}$$

Also, from (43), we have

$$\Phi(s,s,u_1,u_2) = [s^{-1}\Phi(s,s,s,u_1) + (1-s^{-1})\chi(s,s)] \times \Theta[(1-\rho)su_2 + \rho s]\, \Psi(s), \tag{72}$$

and

$$\Phi(s,u_1,u_2,u_3) = [s^{-1}\Phi(s,s,u_1,u_2) + (1-s^{-1})\chi(s,u_1)] \times \Theta[(1-\rho)su_3 + \rho s]\Psi(s). \tag{73}$$

From (71) to (73), it follows that

$$\Phi(s,u_1,u_2,u_3) = \frac{(1-s)}{s}\Theta[(1-\rho)su_3 + \rho s]\Psi(s) \times \left\{\Theta[(1-\rho)su_2 + \rho s]\frac{\Psi(s)}{s} \times \left[\frac{\Theta[(1-\rho)su_1 + \rho s]\Psi(s)}{\{\Theta[(1-\rho)s^2 + \rho s]\Psi(s) - s\}} - 1\right]\chi(s,s) - \chi(s,u_1)\right\}. \tag{74}$$

From finiteness at $s = 0$ we deduce, with the help of (6), that

$$\chi(0,u_1) = \mu[1 + (1-\rho)p_1q_0u_1]c_{000}. \tag{75}$$

Hence, from (68),

$$c_{011} = (1 - \rho)p_1q_0c_{000}. \tag{76}$$

If we now let $s \to 0$ in (74), and equate coefficients, we obtain, in addition to (76), the relations

$$c_{122} = (1 - \rho)^2p_1^2q_0^2c_{000},\ c_{222} = (1 - \rho)^2p_0p_2q_0^2c_{000}, \tag{77}$$

and

$$\begin{aligned} c_{111} = (1 - \rho)q_0(p_1 + p_0p_1q_1 + 2\rho p_0p_2q_0)c_{000} \\ + (1 - \rho)p_0p_1q_0^2(c_{111} + c_{011}) - p_0q_0c_{122}. \end{aligned} \tag{78}$$

If we substitute the values of $c_{011}$ and $c_{122}$, from (76) and (77), into (78), we obtain (9).

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